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ON THREE-DIMENSIONAL COMMUTATIVE ALGEBRAS CONSTRUCTED BY MEANS OF TWO-DIMENSIONAL ALGEBRAS



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Abstract. This paper is part of a series of studies devoted to the classification, up to isomorphism, of three-dimensional commutative algebras over an arbitrary base field. The complete classification of finite-dimensional algebras is one of the central problems of algebra theory, yet in the general case it is practically intractable, which is why small dimensions are treated separately. The aim of the study is to construct the list of all three-dimensional commutative algebras containing the two-dimensional algebra $A_4(\beta_1, 1)$ as a subalgebra over an arbitrary field F satisfying $\text{Char}(F) \neq 2, 3$ and $F^2 = F$. The work employs the apparatus of the matrix of structure constants (MSC) together with the matrix form of isomorphism $B = g^{-1}Ag^{\otimes 2}$. The equality $\text{Aut}(A_4(\beta_1, 1)) = \{I\}$ substantially narrows the group of basis changes preserving the chosen subalgebra and makes this case essentially different from the previously studied $A_3(\alpha_1, \alpha_4, 1 - \alpha_1)$. By means of a successive normalisation of the parameters and a case-by-case analysis, 30 families of algebras $C_1, \dots, C_{30}$ are obtained. The scientific novelty of the work is that such a complete list for the subalgebra $A_4(\beta_1, 1)$ is presented for the first time. The result obtained serves as a basis for the next stage of the classification - the elimination of redundant families by means of isomorphism invariants.

Keywords: three-dimensional algebra, commutative algebra, subalgebra, isomorphism, automorphism group, matrix of structure constants, classification, base field, canonical form.

Annotatsiya. Ushbu maqola ixtiyoriy asosiy maydon ustidagi uch o'lchovli kommutativ algebralarni izomorfizmga aniqlikda tasniflashga bag'ishlangan tadqiqotlar turkumining bir qismi hisoblanadi. Chekli o'lchovli algebralarni to'liq tasniflash algebra nazariyasining markaziy masalalaridan biri bo'lib, umumiy holda uni yechish amalda deyarli imkonsiz, shu bois kichik o'lchamli hollar alohida o'rganiladi. Tadqiqotning maqsadi - $\text{Char}(F) \neq 2, 3$ and $F^2 = F$ shartlarini qanoatlantiruvchi ixtiyoriy F maydon ustida ikki o'lchovli $A_4(\beta_1, 1)$ algebrani qism algebra sifatida o'z ichiga oluvchi barcha uch o'lchovli kommutativ algebralarni ro'yxatini qurishdan iborat. Ishda tizimiy konstantalar matritsasi (TKM) apparati va izomorfizmning $B = g^{-1}Ag^{\otimes 2}$ ko'rinishidagi matritsaviy talqinidan foydalanilgan. $\text{Aut}(A_4(\beta_1, 1)) = \{I\}$ tenglik tanlangan qism algebrani saqlovchi bazis almashtirishlar guruhini sezilarli darajada toraytiradi va bu holni ilgari qaralgan $A_3(\alpha_1, \alpha_4, 1 - \alpha_1)$ holdan tubdan farqlantiradi. Parametrlarni bosqichma-bosqich normallashtirish va hollarga ajratish usuli yordamida jami 30 ta algebra oilasi $C_1, \dots, C_{30}$ aniqlandi. Ishning ilmiy yangiligi shundan iboratki, $A_4(\beta_1, 1)$ qism algebra uchun bunday to'liq ro'yxat birinchi marta keltirilmoqda. Olingan natija tasniflashning keyingi bosqichi - izomorfizm invariantlari yordamida ortiqcha oilalarni yo'qotish uchun asos bo'lib xizmat qiladi.

Kalit so'zlar: uch o'lchovli algebra, kommutativ algebra, qism algebra, izomorfizm, avtomorfizmlar guruhi, tizimiy konstantalar matritsasi, tasniflash, asosiy maydon, kanonik ko'rinish.

Аннотация. Настоящая статья является частью цикла исследований, посвящённых классификации трёхмерных коммутативных алгебр над произвольным основным полем с точностью до изоморфизма. Полная классификация конечномерных алгебр - одна из центральных задач теории алгебр, однако в общем случае она практически неразрешима, поэтому малые размерности изучаются отдельно. Цель исследования - построить список всех трёхмерных коммутативных алгебр, содержащих двумерную алгебру $A_4(\beta_1, 1)$ в качестве подалгебры, над произвольным полем F , удовлетворяющим условиям $\text{Char}(F) \neq 2, 3$ and $F^2 = F$. В работе используется аппарат матрицы структурных констант (МСК) и матричная трактовка изоморфизма вида $B = g^{-1}Ag^{\otimes 2}$. Равенство $\text{Aut}(A_4(\beta_1, 1)) = \{I\}$ существенно сужает группу замен базиса, сохраняющих выбранную подалгебру, и принципиально отличает данный случай от ранее рассмотренного $A_3(\alpha_1, \alpha_4, 1 - \alpha_1)$. С помощью последовательной нормализации параметров и разбора случаев получено 30 семейств алгебр $C_1, \dots, C_{30}$. Научная новизна работы состоит в том, что такой полный список для подалгебры $A_4(\beta_1, 1)$ приводится впервые. Полученный результат служит основой для следующего этапа классификации - устранения дублирующих семейств с помощью инвариантов изоморфизма

Ключевые слова: трёхмерная алгебра, коммутативная алгебра, подалгебра, изоморфизм, группа автоморфизмов, матрица структурных констант, классификация, основное поле, каноническая форма.

INTRODUCTION

The classification of finite-dimensional algebras up to isomorphism is one of the fundamental problems of algebra theory, but solving it completely in the general case is practically impossible. For this reason, researchers usually restrict themselves to algebras of small dimension: they construct candidate structures on the basis of known algebras of lower dimension and then decide isomorphism between them by means of structural invariants - the automorphism group, the derivation algebra and the like. Similar approaches are also being applied in recent studies devoted to the classification of algebras over an arbitrary field. The topicality of the subject lies in the fact that, although the complete classification of two-dimensional algebras is already known, the final list for the three-dimensional case over an arbitrary field has still not been constructed; this calls for a consistent and verifiable method of passing from the lower dimension to the higher one.

The object of the study is the three-dimensional commutative algebras over a base field F . The subject of the study is the structure constants of those three-dimensional commutative algebras that contain the two-dimensional algebra $A_4(\beta_1, 1)$ as a subalgebra, together with their canonical forms. The aim of the study is to construct the complete list of such algebras under the conditions $\text{Char}(F) \neq 2, 3$ and $F^2 = F$. Proceeding from this aim, the following tasks are set: (i) to write down the general form of the MSC of the three-dimensional commutative algebras preserving the subalgebra $A_4(\beta_1, 1)$; (ii) to describe the group of basis changes that leave this subalgebra invariant; (iii) under the action of that group, to normalise the free parameters step by step and to examine all possible cases exhaustively; (iv) to present the canonical MSC corresponding to each case together with its multiplication table.

The present paper constitutes the first stage of a series of studies aimed at classifying the three-dimensional commutative algebras over a base field F satisfying $F = F^2$. The general idea of the series is the following: to construct and classify the structure of three-dimensional commutative algebras step by step, starting from a known two-dimensional subalgebra of theirs. Recently, in [3], the subalgebra $A_3(\alpha_1, \alpha_4, 1 - \alpha_1)$ was taken from the list of two-dimensional algebras given in [1] and all the corresponding three-dimensional extensions were constructed on that basis. In the present paper we consider the commutative case $A_4(\beta_1, 1)$ of the family $A_4(\beta_1, \beta_2)$ from the list of [1]. Its essential difference is that, according to [2], $\text{Aut}(A_4(\beta_1, \beta_2)) = \{I\}$. This difference between the automorphism groups is not accidental: it is precisely the elements of this group that are used when canonical forms are constructed, and therefore whether it is trivial or non-trivial has a considerable effect on the number and on the shape of the isomorphism classes obtained. For this reason, it is expedient to present the cases A_3 and A_4 as two separate, independent results.

In this paper we construct all three-dimensional commutative algebras containing $A_4(\beta_1, 1)$ as a subalgebra, under the conditions $\text{Char}(F) \neq 2, 3$ and $F^2 = F$ on the base field F .

LITERATURE REVIEW

In recent years, the problem of classifying low-dimensional algebras over an arbitrary base field has taken shape as a separate line of research. One of the principal results in this direction is the work [1] of U. Bekbaev, in which the complete classification of all two-dimensional algebras over an arbitrary base field is given. The author relies on the idea of identifying an algebra with its matrix of structure constants and of studying the orbits of the action $\rho(g, A) = gA(g^{-1})^{\otimes 2}$ of the group $\text{GL}_n(F)$ on the space $M_{n \times n^2}(F)$; as a result, the two-dimensional algebras are listed in the form of several parametric families - among them the family $A_4(\beta_1, \beta_2)$ that is of interest to us. That list serves as the starting material for the present paper.

In the work [2] of Sh. Eshmirzayev and U. Bekbaev, the automorphism group of every two-dimensional algebra in that list is described completely. It is precisely this work that is of decisive importance for our study, since it yields $\text{Aut}(A_4(\beta_1, \beta_2)) = \{I\}$. The triviality of the automorphism group, on the one hand, reduces the freedom available for reduction to a canonical form and, on the other hand, simplifies the tree of cases - because the transformations preserving the subspace are then confined to those that change e_3 only.

In the work [3] of I. Rakhimov and M. Arzikulova, the scheme described above was carried out for the two-dimensional subalgebra $A_3(\alpha_1, \alpha_4, 1 - \alpha_1)$, and all three-dimensional commutative extensions corresponding to it were constructed. What distinguishes the present paper from [3] is that here the subalgebra taken is $A_4(\beta_1, 1)$, whose automorphism group is trivial. As noted above, this is not a minor technical difference: whereas in the case A_3 the non-trivial automorphisms of the subalgebra were used to eliminate superfluous parameters, here the whole burden of normalisation falls on the transformations g that change the basis vector e_3 alone. As a consequence, the tree of cases acquires a different structure and the final list turns out to have a different shape

as well. Thus, although the list of two-dimensional algebras [1] and their automorphisms [2] are fully known in the existing literature, the list of three-dimensional extensions based on the subalgebra $A_4(\beta_1, 1)$ has not been constructed so far; the present paper is aimed precisely at filling this gap.

RESEARCH METHODOLOGY

The study uses the method of representing algebras by their matrix of structure constants and of interpreting the isomorphism relation as a matrix equation. The content of the method is as follows.

Let V be an n -dimensional vector space over a field F . An algebra system is introduced on this vector space by means of a map $\cdot: V \times V \rightarrow V$, $(x, y) \mapsto x \cdot y$, satisfying the condition

$$(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z),$$

where $x, y, z \in V$ and $\alpha, \beta \in F$.

Definition 1. Algebras $A = (V, \cdot_A)$ and $B = (V, \cdot_B)$ are said to be isomorphic if there exists an invertible linear map $f: V \rightarrow V$ such that the equality

$$f(x \cdot_A y) = f(x) \cdot_B f(y)$$

holds, where $x, y \in A$.

Definition 2. An invertible linear map $f: V \rightarrow V$ is called an automorphism if the equality $f(x \cdot y) = f(x) \cdot f(y)$ holds for all $x, y \in A = (V, \cdot)$.

The set of all automorphisms of an algebra A forms a group with respect to composition and is denoted by $\text{Aut}(A)$.

Let $A = (V, \cdot)$ be an n -dimensional algebra over F and let $e = (e_1, e_2, \dots, e_n)$ be its basis. Then the bilinear map $\cdot: V \times V \rightarrow V$ is expressed by means of an $n \times n^2$ matrix (the matrix of structure constants, MSC for short)

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & \dots & a_{1n}^1 & a_{21}^1 & a_{22}^1 & \dots & a_{2n}^1 & \dots & a_{n1}^1 & a_{n2}^1 & \dots & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 & a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 & \dots & a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{11}^n & a_{12}^n & \dots & a_{1n}^n & a_{21}^n & a_{22}^n & \dots & a_{2n}^n & \dots & a_{n1}^n & a_{n2}^n & \dots & a_{nn}^n \end{pmatrix}$$

where the a_{ij}^k are defined as follows:

$$e_i \cdot e_j = \sum_{k=1}^n a_{ij}^k e_k, \quad i, j = 1, 2, \dots, n.$$

If a basis e has been fixed, the algebra A may be regarded as equivalent to its MSC $A \in M_{n \times n^2}(F)$ of size $(n \times n^2)$. Hence from now on we assume the basis to be fixed, we make no distinction between the algebra A and its MSC A , and we work with the matrices A . Commutativity implies $e_i \cdot e_j = e_j \cdot e_i$ for $i \neq j$, that is, the i -th and the j -th columns of the MSC coincide. For this reason $C_n(F)$ may be identified with $MM_{n \times n^2}(F)$.

The product in A is written with respect to the basis e as follows:

$$x \cdot y = eA(x \otimes y),$$

where $x = ex$ and $y = ey$ are arbitrary elements, $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ are the coordinate column vectors of x and y respectively, and $x \otimes y$ is the tensor (Kronecker) product of x and y . From now on, instead of the product " $x \cdot y$ " in A we use the juxtaposition " xy ".

Note that in terms of the MSC, Definition 1 may be rewritten as follows:

$$gB = Ag^{\otimes 2} \Leftrightarrow B = g^{-1}Ag^{\otimes 2},$$

and this gives rise to the following linear representation of $GL_n(F)$ on $M_{n \times n^2}(F)$:

$$\rho: (GL_n(F), M_{n \times n^2}(F)) \rightarrow M_{n \times n^2}(F): \quad \rho(g, A) = gA(g^{-1})^{\otimes 2}.$$

Thus the problem of classifying algebras up to isomorphism reduces to the problem of describing the orbits of the action ρ , one representative - a canonical MSC - being chosen from each orbit. In practice this is carried out in the following stages: first the subalgebra is fixed and the group of transformations g preserving it is determined;

then the product $g^{-1} A^c g^{\otimes 2}$ is computed and, by means of the free parameters (x, y, z) occurring in it, the entries of the matrix are successively turned into zeros or ones. Every normalisation step gives rise to a bifurcation of the form "the expression is zero - the expression is non-zero"; each leaf of the tree thus obtained corresponds to a single family of algebras. The fact that the tree has been examined in full guarantees the completeness of the result. All symbolic computations were carried out with the aid of a computer algebra system and verified by hand.

We now return to the three-dimensional case and let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \end{pmatrix}$$

be the MSC of A. For a commutative algebra A its MSC is written as follows

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_2 & a_5 & a_6 & a_3 & a_6 & a_9 \\ b_1 & b_2 & b_3 & b_2 & b_5 & b_6 & b_3 & b_6 & b_9 \\ c_1 & c_2 & c_3 & c_2 & c_5 & c_6 & c_3 & c_6 & c_9 \end{pmatrix}$$

ANALYSIS AND RESULTS

In this section, the main result of the paper - the complete list of the three-dimensional commutative algebras in question - is presented, and its proof is set out in detail.

Theorem 1. The list of all three-dimensional commutative algebra systems that contain the two-dimensional algebra $A_4(\beta, 1)$ as a subalgebra and that can be constructed over a field F satisfying $\text{Char}(F) \neq 2, 3$ and $F^2 = F$ is given as follows:

1. $\mathbb{C}_1$: $e_1 e_3 = e_1 + \beta_3 e_2 + \gamma_3 e_3$, $e_2 e_3 = \gamma_6 e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + \gamma_9 e_3$, $\beta_1, \alpha_9, \beta_3, \beta_9, \gamma_3, \gamma_6, \gamma_9 \in \mathbb{F}$.
2. $\mathbb{C}_2$: $e_1 e_3 = e_2 + \gamma_3 e_3$, $e_2 e_3 = \gamma_6 e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + \gamma_9 e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_3, \gamma_6, \gamma_9 \in \mathbb{F}$.
3. $\mathbb{C}_3$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = \gamma_6 e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_3, \gamma_6 \in \mathbb{F}$.
4. $\mathbb{C}_4$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = \gamma_6 e_3$, $e_3 e_3 = e_1 + \beta_9 e_2$, $\beta_1, \beta_9, \gamma_3, \gamma_6 \in \mathbb{F}$.
5. $\mathbb{C}_5$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = \gamma_6 e_3$, $e_3 e_3 = e_2$, $\beta_1, \gamma_3, \gamma_6 \in \mathbb{F}$.
6. $\mathbb{C}_6$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = \gamma_6 e_3$, $\beta_1, \gamma_3, \gamma_6 \in \mathbb{F}$.
7. $\mathbb{C}_7$: $e_1 e_3 = e_1 + \gamma_3 e_3$, $e_2 e_3 = \alpha_6 e_1 + e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + \gamma_9 e_3$, $\beta_1, \alpha_6, \alpha_9, \beta_9, \gamma_3, \gamma_9 \in \mathbb{F}$.
8. $\mathbb{C}_8$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = e_1 + e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + \gamma_9 e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_3, \gamma_9 \in \mathbb{F}$.
9. $\mathbb{C}_9$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = e_1 + e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_3 \in \mathbb{F}$.
10. $\mathbb{C}_{10}$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = e_1 + e_3$, $e_3 e_3 = e_1 + \beta_9 e_2$, $\beta_1, \beta_9, \gamma_3 \in \mathbb{F}$.
11. $\mathbb{C}_{11}$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = e_1 + e_3$, $e_3 e_3 = e_2$, $\beta_1, \gamma_3 \in \mathbb{F}$.
12. $\mathbb{C}_{12}$: $e_1 e_3 = \gamma_3 e_3$, $e_2 e_3 = e_3$, $\beta_1, \gamma_3 \in \mathbb{F}$.
13. $\mathbb{C}_{13}$: $e_1 e_3 = \beta_3 e_2 + (2\beta_1 + 1)e_3$, $e_2 e_3 = e_1 + e_3$, $e_3 e_3 = \alpha_9 e_1 + \beta_9 e_2 + \gamma_9 e_3$, $\beta_1, \alpha_9, \beta_3, \beta_9, \gamma_9 \in \mathbb{F}$.

14. $\mathbb{C}_{14}$: $e_1e_3 = e_2 + (2\beta_1 + 1)e_3$, $e_2e_3 = e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2 + \gamma_9e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_9 \in \mathbb{F}$.
15. $\mathbb{C}_{15}$: $e_1e_3 = (2\beta_1 + 1)e_3$, $e_2e_3 = e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2 + e_3$, $\beta_1, \alpha_9, \beta_9 \in \mathbb{F}$.
16. $\mathbb{C}_{16}$: $e_1e_3 = (2\beta_1 + 1)e_3$, $e_2e_3 = e_3$, $e_3e_3 = e_1 + \beta_9e_2$, $\beta_1, \beta_9 \in \mathbb{F}$.
17. $\mathbb{C}_{17}$: $e_1e_3 = (2\beta_1 + 1)e_3$, $e_2e_3 = e_3$, $e_3e_3 = e_2$, $\beta_1 \in \mathbb{F}$.
18. $\mathbb{C}_{18}$: $e_1e_3 = (2\beta_1 + 1)e_3$, $e_2e_3 = e_3$, $\beta_1 \in \mathbb{F}$.
19. $\mathbb{C}_{19}$: $e_1e_1 = -\frac{1}{4}e_2$, $e_1e_3 = e_1 + \beta_3e_2 + \frac{1}{2}e_3$, $e_2e_3 = \alpha_6e_1 + e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2$, $\alpha_6, \alpha_9, \beta_3, \beta_9 \in \mathbb{F}$.
20. $\mathbb{C}_{20}$: $e_1e_1 = -\frac{1}{4}e_2$, $e_1e_3 = e_2 + \frac{1}{2}e_3$, $e_2e_3 = \alpha_6e_1 + e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2$, $\alpha_6, \alpha_9, \beta_9 \in \mathbb{F}$.
21. $\mathbb{C}_{21}$: $e_1e_1 = -\frac{1}{4}e_2$, $e_1e_3 = \frac{1}{2}e_3$, $e_2e_3 = e_1 + e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2$, $\alpha_9, \beta_9 \in \mathbb{F}$.
22. $\mathbb{C}_{22}$: $e_1e_1 = -\frac{1}{4}e_2$, $e_1e_3 = \frac{1}{2}e_3$, $e_2e_3 = e_3$, $e_3e_3 = e_1 + \beta_9e_2$, $\beta_9 \in \mathbb{F}$.
23. $\mathbb{C}_{23}$: $e_1e_1 = -\frac{1}{4}e_2$, $e_1e_3 = \frac{1}{2}e_3$, $e_2e_3 = e_3$, $e_3e_3 = e_2$.
24. $\mathbb{C}_{24}$: $e_1e_1 = -\frac{1}{4}e_2$, $e_1e_3 = \frac{1}{2}e_3$, $e_2e_3 = e_3$.
25. $\mathbb{C}_{25}$: $e_1e_3 = \beta_3e_2 + \gamma_3e_3$, $e_2e_3 = e_1 - e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2 + \gamma_9e_3$, $\beta_1, \alpha_9, \beta_3, \beta_9, \gamma_3, \gamma_9 \in \mathbb{F}$.
26. $\mathbb{C}_{26}$: $e_1e_3 = e_2 + \gamma_3e_3$, $e_2e_3 = -e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2 + \gamma_9e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_3, \gamma_9 \in \mathbb{F}$.
27. $\mathbb{C}_{27}$: $e_1e_3 = \gamma_3e_3$, $e_2e_3 = -e_3$, $e_3e_3 = \alpha_9e_1 + \beta_9e_2 + e_3$, $\beta_1, \alpha_9, \beta_9, \gamma_3 \in \mathbb{F}$.
28. $\mathbb{C}_{28}$: $e_1e_3 = \gamma_3e_3$, $e_2e_3 = -e_3$, $e_3e_3 = e_1 + \beta_9e_2$, $\beta_1, \beta_9, \gamma_3 \in \mathbb{F}$.
29. $\mathbb{C}_{29}$: $e_1e_3 = \gamma_3e_3$, $e_2e_3 = -e_3$, $e_3e_3 = e_2$, $\beta_1, \gamma_3 \in \mathbb{F}$.
30. $\mathbb{C}_{30}$: $e_1e_3 = \gamma_3e_3$, $e_2e_3 = -e_3$, $\beta_1, \gamma_3 \in \mathbb{F}$.

Proof. We carry out the proof in two steps: first, keeping the subalgebra $A_4(\beta_1, 1)$ fixed, we write down the general form of all three-dimensional commutative algebras containing it; then, by means of a basis change (an automorphism), we simplify this general form as far as possible and bring it to a canonical form.

Step 1: the subalgebra and its extension. The complete classification of two-dimensional algebras is given in [1], and their automorphism groups in [2]. The list in [1] contains the family

$$A_4(\beta_1, \beta_2) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}, \quad \beta_1, \beta_2 \in \mathbb{F},$$

for which, according to [2], $\text{Aut}(A_4(\beta_1, \beta_2)) = \{I\}$. This family is commutative when $\beta_2 = 1$, that is, when the relations

$$e_1 e_1 = \beta_1 e_2, \quad e_1 e_2 = e_1 + e_2, \quad e_2 e_2 = -e_2, \quad \beta_1 \in \mathbb{F}$$

hold. It is precisely this commutative case that we denote below by $A_4(\beta_1, 1)$:

$$\mathbf{A}_4(\beta_1, 1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & -1 \end{pmatrix}, \quad \text{Aut}(\mathbf{A}_4(\beta_1, 1)) = \{I\}.$$

We now consider a three-dimensional commutative algebra A whose basis $\{e_1, e_2\}$ spans $A_4(\beta_1, 1)$; in the general case, the products involving its third basis vector e_3 are expressed through the following unknown structure constants:

$$e_1 e_3 = e_3 e_1 = x_3 e_1 + y_3 e_2 + z_3 e_3, \quad e_2 e_3 = e_3 e_2 = x_6 e_1 + y_6 e_2 + z_6 e_3,$$

$$e_3 e_3 = x_9 e_1 + y_9 e_2 + z_9 e_3,$$

where $x_3, y_3, z_3, x_6, y_6, z_6, x_9, y_9, z_9 \in \mathbb{F}$ are arbitrary parameters. Hence the MSC of A (which we

denote by A^c) consists of the known part corresponding to $A_4(\beta_1, 1)$ together with the 9 parameters above.

Step 2: the basis changes preserving the subalgebra. Since $\text{Aut}(A_4(\beta_1, 1)) = \{I\}$, any linear transformation of A that leaves the subspace $\{e_1, e_2\}$ invariant and preserves the structure of $A_4(\beta_1, 1)$ on it must map e_1, e_2 to themselves, and e_3 to itself multiplied by a scalar $z \neq 0$ with an arbitrary linear combination of e_1, e_2 added. In other words, the group of basis changes sought consists exactly of the matrices of the form

$$g = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{pmatrix}, \quad z \neq 0, \quad x, y \in \mathbb{F}$$

Applying the equality $\mathbf{A}_1 \cong \mathbf{A}_2 \Leftrightarrow A_2 = g^{-1} A_1^c g^{\otimes 2}$ and acting by such an element g , we find the canonical form of the MSC:

$$g^{-1} A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & -xz_3 + x_3z + y & 1 & 0 & -xz_6 + x_6z + x & -xz_3 + x_3z + y & -xz_6 + x_6z + x & M_{19} \\ \beta_1 & 1 & \beta_1x - yz_3 + y_3z + y & 1 & -1 & M_{26} & \beta_1x - yz_3 + y_3z + y & M_{26} & M_{29} \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & M_{39} \end{pmatrix}$$

where

$$M_{19} = -2x^2z_3 + ((2x_3 - z_9)z - 2y(-1 + z_6))x + 2x_6yz + x_9z^2,$$

$$M_{26} = -(z_6 + 1)y + y_6z + x,$$

$$M_{29}$$

$$= -(2z_6 + 1)y^2 + ((-2z_3 + 2)x - z(z_9 - 2y_6))y + \beta_1x^2 + 2xy_3z + y_9z^2,$$

$$M_{39} = 2xz_3 + 2yz_6 + zz_9.$$

In order to remove the common expression M_{26} occurring in positions (1,3) and (2,6) of the matrix above, we take $x = (z_6 + 1)y - y_6z$; since this is a free choice of x , it does not change the isomorphism class. Then

$$g^{-1} A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & M_{13} & 1 & 0 & M_{16} & M_{13} & M_{16} & M_{19} \\ \beta_1 & 1 & M_{23} & 1 & -1 & 0 & M_{23} & 0 & M_{29} \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & ((2z_6 + 2)y - 2y_6z)z_3 + 2z_6y + zz_9 \end{pmatrix}$$

$$M_{13} = ((-z_6 - 1)y + y_6 z)z_3 + x_3 z + y,$$

$$M_{16} = ((-1 + z_6)y_6 + x_6)z - yz_6^2 + y,$$

$$M_{19} = -(2(z_3 + 1)z_6 + 2z_3 - 2)(z_6 + 1)y^2 + (2((2z_3 + 1)z_6 + 2z_3 - 1)y_6 + 2\left(x_3 - \frac{1}{2}z_9\right)z_6 + 2x_3 + 2x_6 - z_9)zy - \left(2y_6^2 z_3 + 2\left(x_3 - \frac{1}{2}z_9\right)y_6 - x_9\right)z^2,$$

$$M_{23} = ((z_6 + 1)\beta_1 - z_3 + 1)y - z(\beta_1 y_6 - y_3),$$

$$M_{29} = (\beta_1 z_6^2 + (2\beta_1 - 2z_3)z_6 + \beta_1 - 2z_3 + 1)y^2 - (2(\beta_1 y_6 - y_3)z_6 + 2(\beta_1 - z_3)y_6 - 2y_3 + z_9)zy + z^2(\beta_1 y_6^2 - 2y_3 y_6 + y_9).$$

Case 1. When $z_6^2 \neq 1$, we take $y = \frac{((-1+z_6)y_6+x_6)z}{z_6^2-1}$, we take . As a result, the MSC takes the following

form:

$$M_{13} = ((-z_3 z_6 - z_3 + 1)x_6 + (-1 + z_6)(x_3 z_6 + x_3 + y_6))/(z_6^2 - 1),$$

$$M_{19} = \left(-2x_6^2 z_3 + 2(-1 + z_6)\left(x_3 - \frac{1}{2}z_9\right)x_6 + x_9(-1 + z_6)^2\right)/(-1 + z_6)^2,$$

$$M_{23} = \frac{1}{(z_6^2 - 1)} (y_3 z_6^2 + (\beta_1 x_6 - y_6(z_3 - 1))z_6 + (\beta_1 - z_3 + 1)x_6 + y_6(z_3 - 1) - y_3),$$

$$M_{29} = \frac{1}{(z_6^2 - 1)^2} (y_9 z_6^4 + (2x_6 y_3 - y_6 z_9)z_6^3 + (\beta_1 x_6^2 + (-2y_6 z_3 + 2y_3 - z_9)x_6 + y_6^2 + y_6 z_9 - 2y_9)z_6^2 + ((2\beta_1 - 2z_3)x_6^2 + (-2y_3 + 2y_6)x_6 - 2y_6^2 + y_6 z_9)z_6 + (\beta_1 - 2z_3 + 1)x_6^2 + ((2z_3 - 2)y_6 - 2y_3 + z_9)x_6 + y_6^2 - y_6 z_9 + y_9),$$

$$M_{39} = \left(2\left(y_6 + \frac{1}{2}z_9\right)z_6^2 + 2((z_3 + 1)x_6 - y_6)z_6 + 2z_3 x_6 - z_9\right)/(z_6^2 - 1).$$

Case 11. When $M_{13} \neq 0$, we take $z = \frac{1}{M_{13}}$. The MSC thereby reduces to the following form:

$$\mathbb{C}_1 = g^{-1} A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & \frac{1}{M_{13}^2} M_{19} \\ \beta_1 & 1 & \frac{1}{M_{13}} M_{23} & 1 & -1 & 0 & \frac{1}{M_{13}} M_{23} & 0 & \frac{1}{M_{13}^2} M_{29} \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & \frac{1}{M_{13}} M_{39} \end{pmatrix}$$

Case 121. When $M_{13} = 0$ and $M_{23} \neq 0$, we take $z = \frac{1}{M_{23}}$. In this case the MSC is as follows:

$$\mathbb{C}_2 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{M_{23}^2}M_{19} \\ \beta_1 & 1 & 1 & 1 & -1 & 0 & 1 & 0 & \frac{1}{M_{23}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & \frac{1}{M_{23}}M_{39} \end{pmatrix}$$

Case 1221. When $M_{13} = 0$ and $M_{23} = 0$ and $M_{39} \neq 0$, we take $z = \frac{1}{M_{39}}$.

This yields the following MSC:

$$\mathbb{C}_3 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{19} \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & 1 \end{pmatrix}$$

Case 12221. When $M_{13} = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} \neq 0$, we take $z^2 = \frac{1}{M_{19}}$. As a result, the MSC takes the following form:

$$\mathbb{C}_4 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{19}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & 0 \end{pmatrix}$$

Case 122221. When $M_{13} = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} \neq 0$, we take $z^2 = \frac{1}{M_{29}}$. The MSC thereby reduces to the following form:

$$\mathbb{C}_5 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & 0 \end{pmatrix}$$

Case 122222. We now consider the case $M_{13} = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} = 0$; in this case the MSC is as follows:

$$\mathbb{C}_6 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & z_6 & z_3 & z_6 & 0 \end{pmatrix}$$

Case 21. We consider the case $z_6 = 1$.

M_{19}

$$\begin{aligned} &= (-2y_6^2z_3 + (-2x_3 + z_9)y_6 + x_9)z^2 + (8y_6z_3 + 4x_3 + 2x_6 - 2z_9)yz \\ &\quad - 8y^2z_3, \end{aligned}$$

$$M_{29}$$

$$= (4\beta_1 - 4z_3 + 1)y^2 - \left(4\left(\beta_1 - \frac{1}{2}z_3\right)y_6 - 4y_3 + z_9\right)zy + z^2(\beta_1 y_6^2 - 2y_3 y_6 + y_9),$$

$$M_{39} = (-2y_6 z + 4y)z_3 + z z_9 + 2y.$$

Case 211. When $2\beta_1 - z_3 + 1 \neq 0$, we take $y = \frac{z(\beta_1 y_6 - y_3)}{2\beta_1 - z_3 + 1}$. In this case the MSC is as follows:

$$M_{13} = 2z\left(-\frac{1}{2}y_6 z_3^2 + \left(-\frac{1}{2}x_3 + y_3 + \frac{1}{2}y_6\right)z_3 + \left(x_3 + \frac{1}{2}y_6\right)\beta_1 + \frac{1}{2}x_3 - \frac{1}{2}y_3\right)/(2\beta_1 - z_3 + 1),$$

$$M_{19} = \frac{1}{(2\beta_1 - z_3 + 1)^2} \left(-2y_6^2 z_3^3 + 4\left(y_6^2 + \left(-\frac{1}{2}x_3 + 2y_3 + \frac{1}{4}z_9\right)y_6 + \frac{1}{4}x_9\right)z_3^2 + 4\left(-\frac{1}{2}y_6^2 + \left(x_3 - \frac{1}{2}x_6 - \frac{1}{2}z_9\right)\beta_1 + x_3 - \frac{1}{2}z_9 - 2y_3\right)y_6 - x_9\beta_1 - 2y_3^2 + \left(x_3 + \frac{1}{2}x_6 - \frac{1}{2}z_9\right)y_3 - \frac{1}{2}x_9 z_3 + 4\left(\beta_1 x_6 - x_3 + \frac{1}{2}z_9\right)y_6 + x_9\beta_1 + (-2x_3 - x_6 + z_9)y_3 + \frac{1}{2}x_9\left(\beta_1 + \frac{1}{2}\right)\right),$$

$$M_{29} = \frac{1}{(2\beta_1 - z_3 + 1)^2} \left((y_6^2 - 2y_6 z_9 + 4y_9)\beta_1^2 + ((-z_3^2 + 1)y_6^2 + ((4z_3 - 2)y_3 + z_9(z_3 - 1))y_6 - 4y_3^2 + 2y_3 z_9 - 4y_9(z_3 - 1))\beta_1 + 2y_3(z_3 - 1)y_6 - 3y_3^2 - z_9(z_3 - 1)y_3 + y_9(z_3 - 1)^2\right),$$

$$M_{39}$$

$$= 2\left(y_6 z_3^2 + \left(-2y_3 - y_6 - \frac{1}{2}z_9\right)z_3 + (y_6 + z_9)\beta_1 - y_3 + \frac{1}{2}z_9\right)/(2\beta_1 - z_3 + 1).$$

Case 2111. When $M_{13} \neq 0$, we take $z = \frac{1}{M_{13}}$. This yields the following MSC:

$$C_7 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & \frac{1}{M_{13}}x_6 & 1 & \frac{1}{M_{13}}x_6 & \frac{1}{M_{13}^2}M_{19} \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{13}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & 1 & z_3 & 1 & \frac{1}{M_{13}}M_{39} \end{pmatrix}$$

Case 21121. When $M_{13} = 0$ and $x_6 \neq 0$, we take $z = \frac{1}{x_6}$. As a result, the MSC takes the following form:

$$\mathbb{C}_8 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{x_6^2}M_{19} \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{x_6^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & 1 & z_3 & 1 & \frac{1}{x_6}M_{39} \end{pmatrix}$$

Case 211221. When $M_{13} = 0$ and $x_6 = 0$ and $M_{39} \neq 0$, we take $z = \frac{1}{M_{39}}$.

The MSC thereby reduces to the following form:

$$\mathbb{C}_9 = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{M_{39}^2}M_{19} \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & 1 & z_3 & 1 & 1 \end{pmatrix}$$

Case 2112221. When $M_{13} = 0$ and $x_6 = 0$ and $M_{39} = 0$ and $M_{19} \neq 0$, we take $z^2 = \frac{1}{M_{19}}$. In this case the MSC is as follows:

$$\mathbb{C}_{10} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{19}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & 1 & z_3 & 1 & 0 \end{pmatrix}$$

Case 21122221. When $M_{13} = 0$ and $x_6 = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} \neq 0$, we take $z^2 = \frac{1}{M_{29}}$. This yields the following MSC:

$$\mathbb{C}_{11} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & z_3 & 0 & 0 & 1 & z_3 & 1 & 0 \end{pmatrix}$$

Case 21122222. At the next step we deal with the case $M_{13} = 0$ and $x_6 = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} = 0$; in this case the MSC is as follows:

$$\mathbb{C}_{12} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & 1 & z_3 & 1 & 0 \end{pmatrix}$$

Case 212. We take $2\beta_1 - z_3 + 1 = 0$.

$$M_{13} = -y + x_3z - 4\beta_1y + 2\beta_1y_6z + y_6z,$$

$$M_{19} = -8y^2 + 8yy_6z + 4x_3zy - 2x_3zy_6z - 16\beta_1y^2 + 16\beta_1yy_6z - 4\beta_1y_6^2z^2 - 2y_6^2z^2 + 2yx_6z + x_9z^2 - 2zyz_9 + zz_9y_6z,$$

$$M_{23} = -\beta_1y_6z + y_3z,$$

$$M_{29} = -4\beta_1y^2 + \beta_1y_6^2z^2 - 3y^2 + 4y_3zy - 2y_3zy_6z + 2yy_6z + y_9z^2 - zyz_9,$$

$$M_{39} = 8\beta_1y - 4\beta_1y_6z + 6y - 2y_6z + zz_9.$$

Case 2121. When $1 + 4\beta_1 \neq 0$, we take $y = \frac{(x_3+y_6+2\beta_1y_6)z}{1+4\beta_1}$. As a result, the MSC takes the following form:

$$M_{19} = \frac{1}{(1+4\beta_1)^2} \left(-8(2\beta_1+1)(2\beta_1y_6+x_3+y_6)^2 + (4\beta_1+1)^2(-4\beta_1y_6^2-2x_3y_6+x_9-2y_6^2+y_6z_9) + 2(4\beta_1+1)(2\beta_1y_6+x_3+y_6)(8\beta_1y_6+2x_3+x_6+4y_6-z_9) \right),$$

$$M_{23} = -\beta_1y_6 + y_3,$$

$$M_{29} = \frac{1}{(1+4\beta_1)^2} \left((4\beta_1+1)^2(\beta_1y_6^2-2y_3y_6+y_9) + (4\beta_1+1)(4y_3+2y_6-z_9)(2\beta_1y_6+x_3+y_6) - (4\beta_1+3)(2\beta_1y_6+x_3+y_6)^2 \right),$$

$$M_{39} = \frac{1}{1+4\beta_1} \left(8\beta_1x_3 + 8\beta_1y_6 + 16\beta_1^2y_6 - 12\beta_1y_6 - 16\beta_1^2y_6 + 6x_3 + 6y_6 + 12\beta_1y_6 - 2y_6 + z_9 + 4z_9\beta_1 \right).$$

Case 21211. When $x_6 \neq 0$, we take $z = \frac{1}{x_6}$. The MSC thereby reduces to the following form:

$$\mathbb{C}_{13} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{x_6^2}M_{19} \\ \beta_1 & 1 & \frac{1}{x_6}M_{23} & 1 & -1 & 0 & \frac{1}{x_6}M_{23} & 0 & \frac{1}{x_6^2}M_{29} \\ 0 & 0 & 2\beta_1+1 & 0 & 0 & 1 & 2\beta_1+1 & 1 & \frac{1}{x_6}M_{39} \end{pmatrix}$$

Case 212121. When $x_6 = 0$ and $M_{23} \neq 0$, we take $z = \frac{1}{M_{23}}$. In this case the MSC is as follows:

$$\mathbb{C}_{14} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{M_{23}^2}M_{19} \\ \beta_1 & 1 & 1 & 1 & -1 & 0 & 1 & 0 & \frac{1}{M_{23}^2}M_{29} \\ 0 & 0 & 2\beta_1+1 & 0 & 0 & 1 & 2\beta_1+1 & 1 & \frac{1}{M_{23}}M_{39} \end{pmatrix}$$

Case 2121221. When $x_6 = 0$ and $M_{23} = 0$ and $M_{39} \neq 0$, we take $z = \frac{1}{M_{39}}$. This yields the following MSC:

$$C_{15} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{19} \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{29} \\ 0 & 0 & 2\beta_1 + 1 & 0 & 0 & 1 & 2\beta_1 + 1 & 1 & 1 \end{pmatrix}$$

Case 21212221. When $x_6 = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} \neq 0$, we take $z^2 = \frac{1}{M_{19}}$. As a result, the MSC takes the following form:

$$C_{16} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{19}^2}M_{29} \\ 0 & 0 & 2\beta_1 + 1 & 0 & 0 & 1 & 2\beta_1 + 1 & 1 & 0 \end{pmatrix}$$

Case 212122221. When $x_6 = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} \neq 0$, we take $z^2 = \frac{1}{M_{29}}$. The MSC thereby reduces to the following form:

$$C_{17} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2\beta_1 + 1 & 0 & 0 & 1 & 2\beta_1 + 1 & 1 & 0 \end{pmatrix}$$

Case 212122222. We now consider the case $x_6 = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} = 0$; in this case the MSC is as follows:

$$C_{18} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta_1 + 1 & 0 & 0 & 1 & 2\beta_1 + 1 & 1 & 0 \end{pmatrix}$$

Case 2122. This time we consider the case $1 + 4\beta_1 = 0$.

$$M_{13} = x_3 + \frac{1}{2}y_6,$$

$$M_{19} = -4y^2 + 4yy_6z + 4x_3zy - 2x_3zy_6z - y_6^2z^2 + 2yx_6z + x_9z^2 - 2zyz_9 + zz_9y_6z,$$

$$M_{23} = y_3 + \frac{1}{4}y_6,$$

$$M_{29} = -2y^2 - \frac{1}{4}y_6^2z^2 + 4y_3zy - 2y_3zy_6z + 2yy_6z + y_9z^2 - zyz_9,$$

$$M_{39} = 4y - y_6z + zz_9.$$

In the same way, in order to make the expression M_{39} vanish, we take

$$y = \frac{(y_6 - z_9)z}{4};$$

$$g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & zM_{13} & 1 & 0 & x_6z & zM_{13} & x_6z & z^2M_{19} \\ -\frac{1}{4} & 1 & zM_{23} & 1 & -1 & 0 & zM_{23} & 0 & z^2M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

$$M_{13} = x_3 + \frac{1}{2}y_6,$$

$$M_{23} = \frac{1}{4}y_6 + y_3,$$

$$M_{19} = -x_3y_6 - x_3z_9 + \frac{1}{2}x_6y_6 - \frac{1}{2}x_6z_9 + x_9 - \frac{1}{4}y_6^2 + \frac{1}{4}z_9^2,$$

$$M_{29} = -y_3y_6 - y_3z_9 + \frac{1}{8}y_6^2 - \frac{1}{2}y_6z_9 + y_9 + \frac{1}{8}z_9^2.$$

Case 21221. When $M_{13} \neq 0$, we take $z = \frac{1}{M_{13}}$. In this case the MSC is as follows:

$$\begin{aligned} \mathbb{C}_{19} &= g^{-1}A^c g^{\otimes 2} \\ &= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & \frac{1}{M_{13}}x_6 & 1 & \frac{1}{M_{13}}x_6 & \frac{1}{M_{13}^2}M_{19} \\ -\frac{1}{4} & 1 & \frac{1}{M_{13}}M_{23} & 1 & -1 & 0 & \frac{1}{M_{13}}M_{23} & 0 & \frac{1}{M_{13}^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix} \end{aligned}$$

Case 212221. When $M_{13} = 0$ and $M_{23} \neq 0$, we take $z = \frac{1}{M_{23}}$. This yields the following MSC:

$$\mathbb{C}_{20} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \frac{1}{M_{23}}x_6 & 0 & \frac{1}{M_{23}}x_6 & \frac{1}{M_{23}^2}M_{19} \\ -\frac{1}{4} & 1 & 1 & 1 & -1 & 0 & 1 & 0 & \frac{1}{M_{23}^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 2122221. When $M_{13} = 0$ and $M_{23} = 0$ and $x_6 \neq 0$, we take $z = \frac{1}{x_6}$.

As a result, the MSC takes the following form:

$$C_{21} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{x_6^2}M_{19} \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{x_6^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 21222221. When $M_{13} = 0$ and $M_{23} = 0$ and $x_6 = 0$ and $M_{19} \neq 0$, we take $z^2 = \frac{1}{M_{19}}$. The MSC thereby reduces to the following form:

$$C_{22} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{19}^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 212222221. When $M_{13} = 0$ and $M_{23} = 0$ and $x_6 = 0$ and $M_{19} = 0$ and $M_{29} \neq 0$, we take $z^2 = \frac{1}{M_{29}}$. In this case the MSC is as follows:

$$C_{23} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 212222222. At the next step we deal with the case $M_{13} = 0$ and $M_{23} = 0$ and $x_6 = 0$ and $M_{19} = 0$ and $M_{29} = 0$; in this case the MSC is as follows:

$$C_{24} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 22. As the next case we take $z_6 = -1$.

In the same way, in order to make the expression M_{39} vanish, we take

$$y = \frac{(y_6 - z_9)z}{4}.$$

$$g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & zM_{13} & 1 & 0 & x_6z & zM_{13} & x_6z & z^2M_{19} \\ -\frac{1}{4} & 1 & zM_{23} & 1 & -1 & 0 & zM_{23} & 0 & z^2M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

$$M_{13} = x_3 + \frac{1}{2}y_6,$$

$$M_{23} = \frac{1}{4}y_6 + y_3,$$

$$M_{19} = -x_3y_6 - x_3z_9 + \frac{1}{2}x_6y_6 - \frac{1}{2}x_6z_9 + x_9 - \frac{1}{4}y_6^2 + \frac{1}{4}z_9^2,$$

$$M_{29} = -y_3y_6 - y_3z_9 + \frac{1}{8}y_6^2 - \frac{1}{2}y_6z_9 + y_9 + \frac{1}{8}z_9^2.$$

Case 21221. When $M_{13} \neq 0$, we take $z = \frac{1}{M_{13}}$. In this case the MSC is as follows:

$$\begin{aligned} \mathbb{C}_{19} &= g^{-1}A^c g^{\otimes 2} \\ &= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & \frac{1}{M_{13}}x_6 & 1 & \frac{1}{M_{13}}x_6 & \frac{1}{M_{13}^2}M_{19} \\ -\frac{1}{4} & 1 & \frac{1}{M_{13}}M_{23} & 1 & -1 & 0 & \frac{1}{M_{13}}M_{23} & 0 & \frac{1}{M_{13}^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix} \end{aligned}$$

Case 212221. When $M_{13} = 0$ and $M_{23} \neq 0$, we take $z = \frac{1}{M_{23}}$. This yields the following MSC:

$$\mathbb{C}_{20} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \frac{1}{M_{23}}x_6 & 0 & \frac{1}{M_{23}}x_6 & \frac{1}{M_{23}^2}M_{19} \\ -\frac{1}{4} & 1 & 1 & 1 & -1 & 0 & 1 & 0 & \frac{1}{M_{23}^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 21222221. When $M_{13} = 0$ and $M_{23} = 0$ and $x_6 \neq 0$, we take $z = \frac{1}{x_6}$.

As a result, the MSC takes the following form:

$$\mathbb{C}_{21} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{x_6^2}M_{19} \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{x_6^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 21222221. When $M_{13} = 0$ and $M_{23} = 0$ and $x_6 = 0$ and $M_{19} \neq 0$, we take $z^2 = \frac{1}{M_{19}}$. The MSC thereby reduces to the following form:

$$\mathbb{C}_{22} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{19}^2}M_{29} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 21222221. When $M_{13} = 0$ and $M_{23} = 0$ and $x_6 = 0$ and $M_{19} = 0$ and $M_{29} \neq 0$, we take $z^2 = \frac{1}{M_{29}}$. In this case the MSC is as follows:

$$\mathbb{C}_{23} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 21222222. At the next step we deal with the case $M_{13} = 0$ and $M_{23} = 0$ and $x_6 = 0$ and $M_{19} = 0$ and $M_{29} = 0$; in this case the MSC is as follows:

$$\mathbb{C}_{24} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Case 22. As the next case we take $z_6 = -1$.

$$M_{19} = 2 \left(\left(-y_6^2 z_3 + \left(-x_3 + \frac{1}{2} z_9 \right) y_6 + \frac{1}{2} x_9 \right) z + y(x_6 - 2y_6) \right) z,$$

$$M_{29} = z^2 (\beta_1 y_6^2 - 2y_3 y_6 + y_9) + (2y_6 z_3 - z_9) y z + y^2.$$

For $y = -(y_6 z_3 + x_3) z$:

$$g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & zM_{16} & 0 & zM_{16} & z^2M_{19} \\ \beta_1 & 1 & zM_{23} & 1 & -1 & 0 & zM_{23} & 0 & z^2M_{29} \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & zM_{39} \end{pmatrix}$$

$$M_{23} = -((-z_3^2 + \beta_1 + z_3)y_6 - x_3z_3 + x_3 - y_3),$$

$$M_{16} = x_6 - 2y_6,$$

$$M_{19} = -(2x_6 - 2y_6)(y_6z_3 + x_3) + (y_6z_9 + x_9),$$

$$M_{29} = (-z_3^2 + \beta_1)y_6^2 + (z_3z_9 - 2y_3)y_6 + x_3^2 + z_9x_3 + y_9,$$

$$M_{39} = 2x_3 + z_9.$$

Case 221. When $M_{16} \neq 0$, we take $z = \frac{1}{M_{16}}$. This yields the following MSC:

$$\mathbb{C}_{25} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{M_{16}^2}M_{19} \\ \beta_1 & 1 & \frac{1}{M_{16}}M_{23} & 1 & -1 & 0 & \frac{1}{M_{16}}M_{23} & 0 & \frac{1}{M_{16}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & \frac{1}{M_{16}}M_{39} \end{pmatrix}$$

Case 2221. When $M_{16} = 0$ and $M_{23} \neq 0$, we take $z = \frac{1}{M_{23}}$. As a result, the MSC takes the following form:

$$\mathbb{C}_{26} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{M_{23}^2}M_{19} \\ \beta_1 & 1 & 1 & 1 & -1 & 0 & 1 & 0 & \frac{1}{M_{23}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & \frac{1}{M_{23}}M_{39} \end{pmatrix}$$

Case 22221. When $M_{16} = 0$ and $M_{23} = 0$ and $M_{39} \neq 0$, we take $z = \frac{1}{M_{39}}$.

The MSC thereby reduces to the following form:

$$\mathbb{C}_{27} = g^{-1}A^c g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{19} \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{39}^2}M_{29} \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & 1 \end{pmatrix}$$

Case 222221. When $M_{16} = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} \neq 0$, we take $z^2 = \frac{1}{M_{19}}$. In this case the MSC is as follows:

$$C_{28} = g^{-1} A^e g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{M_{19}^2} M_{29} \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & 0 \end{pmatrix}$$

Case 2222221. When $M_{16} = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} \neq 0$, we take $z^2 = \frac{1}{M_{29}}$. This yields the following MSC:

$$C_{29} = g^{-1} A^e g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & 0 \end{pmatrix}$$

Case 2222222. We now consider the case $M_{16} = 0$ and $M_{23} = 0$ and $M_{39} = 0$ and $M_{19} = 0$ and $M_{29} = 0$; in this case the MSC is as follows:

$$C_{30} = g^{-1} A^e g^{\otimes 2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & -1 & z_3 & -1 & 0 \end{pmatrix}$$

In this way, all possible cases have been examined exhaustively: at each branching point, the expression under consideration (M_{13} , M_{16} , M_{23} , M_{29} , M_{39} , x_6 , etc.) was assumed either to be equal to zero or to be non-zero, and in

the latter case it was set equal to one by means of the corresponding free parameter (z or z^2). As a result, the 30 leaves of the tree yield precisely the families $C_1, \dots, C_{30}$ listed in Theorem 1, which completes the proof of Theorem 1.

CONCLUSION AND SUGGESTIONS

In this paper, all three-dimensional commutative algebras containing the two-dimensional algebra $A_4(\beta_1, 1)$ as a subalgebra have been constructed over an arbitrary base field F satisfying $\text{Char}(F) \neq 2, 3$ and $F^2 = F$, and their complete list has been presented in the form $C_1, \dots, C_{30}$. The result was obtained by means of the apparatus of the matrix of structure constants and a step-by-step normalisation method based on the isomorphism relation $B = g^{-1} A g^{\otimes 2}$.

The following conclusions were reached in the course of the study. First, since $\text{Aut}(A_4(\beta_1, 1)) = \{I\}$, the group of basis changes preserving the subalgebra is restricted to the triangular matrices g that alter the vector e_3 only; this confines the freedom available for normalisation to three parameters (x, y, z). Second, in spite of this restricted freedom, all branches of the tree of cases were examined in full, and the tree turned out to have exactly 30 leaves. Third, the list obtained differs appreciably in structure from the list constructed earlier in [3] for the subalgebra $A_3(\alpha_1, \alpha_4, 1 - \alpha_1)$, which clearly demonstrates the influence of the automorphism group of the subalgebra on the outcome of the classification.

It should be emphasised that the list presented is the result of the first stage of the classification process, and that in the general case the families in it may be repetitive; that is, some of the algebras in the list may well turn out to be isomorphic to one another. For this reason, the following are proposed for further research: (i) to compute, for the families $C_1, \dots, C_{30}$, isomorphism invariants such as the automorphism groups, the derivation algebras, the existence of an identity element, associativity and the cohomology groups; (ii) using these invariants, to eliminate the redundant families and to form the final list of isomorphism classes; (iii) to apply the same scheme to the remaining families of the list of two-dimensional algebras in [1] and thereby complete the classification of three-dimensional commutative algebras; (iv) to study the cases $\text{Char}(F) = 2$ and $\text{Char}(F) = 3$ separately, since the normalisations in the present work make use of division by 2 and by 3.

REFERENCES

1. Bekbaev U. Classification of two-dimensional algebras over any basic field // AIP Conference Proceedings. - 2023. - Vol. 2880, 030001. - P. 1–9.
2. Eshmirzayev Sh., Bekbaev U. A description of automorphism groups of all two-dimensional algebras over any basic field // arXiv:2409.08814 [math.RA]. - 2024.
3. Rakhimov I., Arzikulova M. On three-dimensional commutative algebras over an arbitrary base field // PGSYM2, UPM. - 2026. - P. 1-39.

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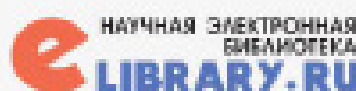
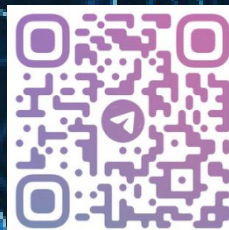
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